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SUMMARY

Methods of automatically controlling the airplane are reviewed. Equations for the controlled motion including inertia effects of the control are developed and methods of investigating the stability of the resulting fifth and higher order equations are presented. The equations for longitudinal and lateral motion with both ideal and non-ideal controls are developed in dimensionless form in terms of control parameters based on simple dynamic tests of the isolated control unit.

INTRODUCTION

Automatic control implies the process of making some physical quantity take on an arbitrary and predetermined series of values without human supervision. A perfect control for aircraft would maintain the airplane along a desired flight path and would completely suppress undesired disturbances in pitching, rolling, and yawing.

The means for applying such a complete constraint to the airplane are lacking. In the conventional airplane, the pilot can influence the motion only by movement of the elevators, the rudder, the ailerons, and the throttle. The law by which these controls are adjusted can be related to any characteristic of the motion, but the controlling influences can be applied only as rolling, yawing, and pitching moments, and as a longitudinal force.

The problem of automatic control lies in relating these controlling influences to the natural characteristics of the airplane so as most nearly to attain the performance of the perfect control.

In order to study the motion of the controlled airplane, it is necessary to extend the equations of motion

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to include the physical characteristics of the control, which in any actual case will not act instantaneously and may possess various kinds of lag. In the following paper, therefore, attention is first given to the performance of a generalized control, isolated from its controlled member, and the complete controlled motion of the airplane is then established in terms of parameters based on the free motion of the control.

STUDY OF THE ISOLATED CONTROL, WITH METHODS FOR DETERMINING THE STABILITY OF HIGHER ORDER EQUATIONS

Control Characteristics

The controls to be discussed are all "error sensitive"; that is, they operate to maintain some quantity constant but derive the impulse for their operation from an error in this quantity. While they can make the error very small, they cannot entirely eliminate it.

Three degrees of sensitivity to the error may be noted. The control force may be a function of the error magnitude, of the rate at which the error is changing, or of the second derivative of the error. In the most general case, the controlling force is proportional to both the error and its derivatives, and may be expressed as

$$F = a_1\epsilon + a_2\dot{\epsilon} + a_3\ddot{\epsilon} \tag{1}$$

Minorsky (reference 1) has suggested that the error and its derivatives might also govern the rate at which the controlling force was applied. The two additional cases that he advances may be written as

$$dF/dt = b_1\epsilon + b_2\dot{\epsilon} + b_3\ddot{\epsilon} \tag{2}$$

$$d^2F/d^2t = c_1\epsilon + c_2\dot{\epsilon} + c_3\ddot{\epsilon} \tag{3}$$

where

ϵ is the error (difference between the value desired and the actual value of the controlled quantity).

F, the controlling force.

a, b, and c, constants of proportionality.

The three types will be called Class I, Class II, and Class III, respectively. It will be seen that Class I controls allow a constant error when there is a steady disturbing force. This error can be reduced by increasing the control sensitivity, but it cannot be completely eliminated. Most automatic controls for airplanes fall in Class I and are usually without benefit of the derivative components (a_2 and a_3 = 0).

Case II controls are used when the controlled quantity is subjected to prolonged and slowly changing disturbances. They allow no steady-state error under constant disturbing influences for as long as an error is present the controlling force increases. Class III controls admit steady-state errors only when the second derivative of the disturbance varies.

The Control as a Simple System

In most cases, the assumption can be made that the control is equivalent to a simple system with only one degree of freedom. The exceptions are the controls in which two of the components have approximately equal frequencies; in this case, the isolated control may develop peculiarities corresponding to a system with more than one degree of freedom. For practical computations, however, the control may be replaced by an equivalent mass, equivalent inertia, equivalent "static stability," and coupling ratios.

The control inherently possesses inertia. The smaller its inertia, the more satisfactory a control is likely to be because the "inertia lag" is reduced. Damping is often added by design to eliminate the tendency of the control-controlled system to hunt. Servo mechanisms embodying hydraulically operated pistons may possess the equivalent of damping because of the resistance of the fluid in the supply lines to change of velocity of flow.

"Static stability" of the control requires that a small departure from the neutral position should produce a force in the control tending to return the control to the neutral position. When the controlled quantity has no inherent stability of its own, the control will generally require static stability. Thus, an azimuth control for airplanes must be statically stable, in order that the airplane - itself insensitive to direction in azimuth - may hold a given course.

It will be assumed that the controls discussed are of the type which Hazen (reference 2) calls "continuous controls." Any error, however small, is considered to produce a corresponding controlling influence through the mechanism. Actual controls may have a small inactive zone within which they are insensitive to errors. The motion of these controls can be determined by solving for the inactive and active regions separately, with due consideration for boundary conditions.

Response of the Isolated Control

The dynamic characteristics of a control isolated from its controlled system can be obtained by subjecting the control to an arbitrary forcing function. The simplest disturbance consists of the sudden application of a constant force or displacement to the control. The theory for these "step functions" will now be developed.

Let

x be the position of some characteristic point on the control, referred to the neutral position of the control.

m, equivalent inertia of the control referred to x.

c, equivalent viscous resistance (damping) of the control referred to x.

k, static stability of the control referred to x.

L, a load suddenly applied to the control.

x_f, a displacement suddenly applied to the control.

Coordinates may, of course, be linear or polar, depending on the physical arrangement of the control.

The equation of motion is, when the force step function is considered,

$$L - m\ddot{x} - c\dot{x} - kx = 0 \tag{4}$$

Then let

ω_n be natural angular frequency of the control $(\sqrt{k/m})$.

ζ = damping ratio, the ratio of the amount of damping present in the control to the amount necessary to produce motion that is just short of being oscillatory. In terms of ζ and ω_n, equation (4) is then

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{L}{k}\,\omega_n^2 \tag{5}$$

and the steady-state displacement of the control is

$$x_{ss} = L/k \tag{6}$$

The general solution for oscillatory motion $(\zeta < 0)$ may be developed as

$$x/x_{ss} = 1 - e^{\zeta\omega_n t}\left(\cos\sqrt{1-\zeta^2}\,\omega_n t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\sqrt{1-\zeta^2}\,\omega_n t\right)$$

If the motion is undamped, this equation simplifies to

$$x/x_{ss} = 1 - \cos\omega_n t \tag{8}$$

and, when $\zeta = 1.00$, the motion is critically damped and is expressed by

$$x/x_{ss} = 1 - e^{-\omega_n t} - \omega_n t^{-\omega_n t} \tag{9}$$

A damping ratio greater than 1.00 produces motion similar to critically damped motion but more sluggish and, as it is essential that the control should operate quickly, the overdamped case will not be of great importance.

Figure 1 illustrates the effect of varying the amount of damping on the response of the control.

When the control is suddenly displaced rather than being disturbed by a force, the equations expressing its return to equilibrium are the same as those presented with the exception that the initial 1.00 is lacking. Thus, if an aperiodic control (equation (9)) is displaced so that its position of equilibrium varies by x_f, the control attains its new equilibrium according to the expression

$$\frac{x}{x_f} = e^{-\omega_n t} - \omega_n t e^{-\omega_n t} \qquad (10)$$

due attention being paid to the proper signs.

Rapidity of response is desired and, since the motion becomes slower with increased damping, it is evident that from this consideration damping should be small. Sufficient damping must be retained, however, to insure that the corresponding oscillation rapidly decays.

Lag in the Control

For most controls it is a straightforward process to set up the equations of controlled motion, assuming that the control has no inertia, dead period, or friction, and acts instantaneously.

It is customary to lump the deficiencies of actual controls that prevent them from attaining this "ideal" performance under the general heading of lag, and to calculate their effects approximately by some semiempirical method.

Strictly speaking, a distinction should be made between the lag of an inactive zone at the neutral position of the control, and the lag that extends over the whole range of operation of the mechanism, such as that caused by the inertia of the parts.

The methods that have been used in treating lag in a control may be grouped roughly under four headings:

1. Introduction of inactive zone of control.

2. Assumption of constant time lag.

3. Use of semiempirical approximations.

4. Use of control characteristics.

Method 1 has been demonstrated by Hazen (reference 2) and Klemin (reference 3). The solution for the motion is obtained in parts, with due consideration for the boundary conditions between regions of active and inactive control.

When lag is known to be present but cannot be exactly evaluated mathematically, as in the case of lag of the human pilot, several writers have used the approximation that the control response lags the error by a constant time interval. Minorsky (reference 1), Callendar, Hartree and Porter (reference 4), and Cowley (reference 5) treat lag by method 2.

Minorsky uses a Taylor's series directly so that, if the controlling influence is a function of conditions m seconds previous, it may be expressed in terms of present time t as

$$f(t-m) = f(t) - mf'(t) + m^2/2! \, f''(t) - \ldots \qquad (11)$$

and if m is small, higher order terms can be neglected.

Callendar, Hartree, and Porter approach the problem by making the usual assumption that the solution of the differential equation is of the form

$$x_t = Ae^{\lambda t} \qquad (12)$$

and then, if the control moves according to x at time $t-m$, the terms expressing the control influence depend on

$$x_{t-m} = Ae^{\lambda(t-m)} \qquad (13)$$

The resulting equation is no longer linear in λ but can be solved by expanding the exponentials in a series and neglecting higher order terms, or by graphical means (reference 4).

Garner (reference 6) has used a simplified method of treating control lag with empirical constants, which amounts to a consideration of the first two terms of either of the foregoing series. The semiempirical approximations of method 3, while useful for a rough check of the effects of control lag, have the difficulty of employing arbitrary constants not always available in any particular case.

Method 4, the introduction of the over-all frequency and the effective damping of the control into the expression for the controlled motion, has been followed in the present paper. This procedure has the advantage of using quantities that can be experimentally measured in the laboratory.

The introduction of additional degrees of freedom into the already complex expression for the motion of the airplane necessitates the development of a method for the treatment of fifth and higher order equations. This method will now be considered.

Higher Order Equations

In order that the motion of the airplane lend itself to mathematical treatment, it is necessary either that all relations be linear or that only small motions be considered. The classical treatment of the motion of the airplane has by now been justified as applicable to disturbances of appreciable magnitude. The linearity of control response depends on the design of the control, but violent movements will not be expected and a linear response for small displacements is a fair assumption.

Solution is obtained by writing the differential equations of motion, assuming a solution of the form

$$x = \Sigma A_k e^{\lambda_k t} \tag{14}$$

and expanding the resulting determinant into an equation linear in λ. The difficulty lies in the solution for the various values of λ from the equation, which is of the form

$$a\lambda^n + b\lambda^{n-1} + c\lambda^{n-2} + d\lambda^{n-3} + \ldots = 0 \tag{15}$$

Methods are available (references 7 to 13) for the solution of the quartic equation and for expressing the complete motion of the uncontrolled airplane. There are also methods of extracting the roots, complex or real, of the quintic, sextic, and higher order equations (references 14, 15) that result from controlled motion; but the methods are long and troublesome, especially when it is desired to investigate a range of possible variations of the control relations.

It will often be sufficient to determine simply the stability of the controlled motion.

Methods of Routh and Hurwitz

Routh (reference 16) presents a series of test functions which can be built up for any degree of linear equation and which indicate from the coefficients of the equation whether the motion it represents will increase or decay with time. The functions are obtained by writing the sequences

$$a \quad b \quad c \quad d \quad e \quad f \quad g \quad . \tag{16}$$

$$b \quad (bc-ad) \quad d \quad (be-af) \quad f \quad (bg-ah) \quad h \tag{17}$$

Beginning with a, each test function is derived from the one preceding it by substituting for each letter in sequence (16), the letter or expression directly below it in sequence (17) (substituting zero for letters above those appearing in the original equation). The motion is then stable if the final function and the coefficients are positive.

Obtained in this manner, the Routhian test function for the quintic is

d(?)

$$\Big[(bc - ad)\ d - b\ (be - af)\Big]\ (be - af) - (bc - ad)^2 f \tag{18}$$

Hurwitz (reference 17) gives a method of obtaining the stability functions as an expansion of the determinant

$$\Delta_H = \begin{vmatrix} b & a & 0 & 0 & 0 & 0 & 0 & . \\ d & c & b & a & 0 & 0 & 0 & . \\ f & e & d & c & b & a & 0 & . \\ h & g & f & e & d & c & b & . \\ . & . & . & . & . & . & . & . \end{vmatrix} \tag{19}$$

The motion is stable if the determinant and the coefficients are positive. This form is somewhat simpler than the one used by Routh for numerical substitution but still involves considerable work for equations above the quartic.

Before a simpler method of following stability changes as the relations of a mechanical system are systematically varied is presented, the form of the solution of the equations of motion will be discussed in more detail.

Form of the Solution

When some of the values of λ_k to be substituted in equation (14) are complex (and therefore conjugate pairs since the coefficients of the original equation were real), there are pairs of roots of the form $a \pm ib$. The part of the solution corresponding to these two roots can then be written

$$x = e^{at} (A_{11} \cos bt + A_{21} \sin bt) \tag{20}$$

The test functions of Routh and Hurwitz, when positive, insure that a be negative, so that the oscillation represented, decays with time. If a is zero, so that λ is a pure imaginary, the term represents an unending oscillation, which is the boundary condition between stability and instability for the term.

The presence of pure imaginary terms is indicated by the fact that Routh's discriminant becomes zero. It should be noted, however, that the discriminant is also zero for more than one pair of equal pure imaginary roots and, when two sets of roots are $\lambda_{1,2} = \lambda_{3,4} = \pm ib$, the motion is unstable, being of the form

$$x = A_1 \cos bt + A_2 \sin bt + A_3 t \cos bt + A_4 t \sin bt \tag{21}$$

Transition to Instability

The fact that a pair of roots becomes pure imaginary as the system from which the equation is derived passes from a stable to an unstable condition is made use of in determining the point of critical stability when some physical characteristic is varied systematically.

Define the <u>angular frequency</u> ω_n of a complete mechanical system of any number of degrees of freedom as the frequency at which it can execute unending oscillation. Simple systems of one degree of freedom will oscillate endlessly only in the absence of damping, but more complicated systems easily and sometimes annoyingly perform self-excited oscillations in spite of a large amount of damping (references 18, 19). The expressions for natural frequency will therefore be established in their most general form, including damping. The most general case of unending oscillation for the systems that can support more than one endless motion will be considered as the case in which the corresponding equation has only one pair of imaginary roots.

Expressions for Natural Frequency

At the boundary of transition from stability to instability, it is known that two of the roots of the equation in λ are $\pm i\omega_n$. Either root can then be substituted back into the original equation, the sum of the real and the imaginary terms be equated to zero, and the reduced equations solved for the frequency. Performed in detail for the cubic equation, the process is as follows:

$$ax^3 + bx^2 + cx + d = 0$$

$$-ia\omega_n^3 - b\omega_n^2 + ci\omega_n + d = 0$$

$$\omega_n^2 = c/a \tag{22.1}$$

$$\omega_n^2 = d/b \tag{22.2}$$

Unending oscillation is indicated when numerical substitution produces the same value of ω_n^2 from both expressions. It is evident that this method is equivalent to setting Routh's discriminant for the cubic $(bc - ad)$ equal to zero and, for this simple case, there is no gain in simplicity by performing the operation in two parts.

For the higher order equations, the pairs of expressions in ω_n are, when similarly obtained,

QUARTIC $\omega_n^2 = d/b$ (23.1)

$$\omega_n^4 = (cd - be)/ba \tag{23.2}$$

QUINTIC $\omega_n^2 = (bc - af)/(bc - ad)$ (24.1)

$$\omega_n^4 = (de - cf)/(bc - ad) \tag{24.2}$$

SEXTIC $$\omega_n^2 = \frac{(bc - ad)f - b^2 g}{(bc - ad)d - b(be - af)} \qquad (25.1)$$

$$\omega_n^4 - (d/b)\,\omega_n^2 + (f/b) = 0 \qquad (25.2)$$

SEPTIC $$\omega_n^2 = \frac{(bc - ad)(fg - eh) - (bg - ah)^2}{(bc - ad)(dg - ch) - (bg - ah)(be - af)} \qquad (26.1)$$

$$(bc - ad)\,\omega_n^4 - (be - af)\,\omega_n^2 + (bg - ah) = 0 \qquad (26.2)$$

The expressions were brought into the form given by straightforward mathematical "juggling." When a single computation of the stability of a mechanical system is desired, Hurwitz's determinant should be numerically expanded, as this method will be found simpler than the several computations necessary to determine, by means of the natural frequencies, whether the system is on the stable or unstable side of the critical point.

When it is desired to observe the transition from stability to instability of a mechanical system, as when studying the effects of control lag or inertia, the frequency method is much more convenient than that of either Routh or Hurwitz. In addition, when the transition point has been determined by this method, the frequency of the endless oscillation is at once available, without further computation.

Application to Sextic Equation

An example of the application of the method to a practical case will now be borrowed from a later section of this report. It is desired to learn within what range of free natural period a longitudinal control of a given sensitivity will be satisfactory, i.e., will not allow self-excited oscillation.

The equation involved is a sextic. It has been determined by applying Routh's discriminant to the quartic equation for the airplane motion under ideal control that the motion is stable when the natural period of the control is infinitely short ($T_N = 0$). As soon as a finite natural period is admitted, the equation of motion becomes a sextic. In figure 2 the two expressions for natural frequency and Routh's discriminant for the sextic have been calculated and plotted against increasing natural period of the control.

By either method, the motion is seen to become unstable when the free natural period of the control exceeds 1.95 seconds, but the actual computations were much simpler for the frequency curves. The <u>frequency</u> of the unending oscillation is, by inspection, about 19 radians per second. It is interesting to note that the <u>frequency</u> of the short and heavily damped oscillation of the ideally controlled airplane considered is 16.9 radians per second.

When the system is nearly critical, it is possible to calculate the rate of growth or decay of the nearly endless oscillation by a method of Blondel's (reference 20) which is based on the assumption that a in the root $a-ib$ is so small that higher powers of a are negligible compared with a itself. The root is then substituted back into the original equation in λ, and real and imaginary terms are separately equated to zero, when solution can be made for a and b.

AUTOMATIC CONTROL OF AIRPLANES

LONGITUDINAL MOTION

If the airplane is slightly disturbed in smooth air and allowed to execute free longitudinal motion, it will, if dynamically stable, regain a steady-flight condition as the disturbed motion decays in the form of two damped oscillations. These two modes of oscillation consist of:

1. A heavily damped oscillation of short period (of the order of a few seconds) involving primarily change of incidence, in which changes of forward velocity are negligible. This motion disappears almost at once, and in most airplanes is not noticeable as an oscillation.

2. A long period, lightly damped oscillation involving change of forward speed, during which the airplane rises and falls. This oscillation depends on the drag of the airplane for its damping, and is increasingly troublesome on "clean" airplanes.

Although the short oscillation has been somewhat neglected in stability analysis, because it disappears so rapidly, Jones (reference 21) points out that, although the heavy damping of this mode insures its rapid subsidence in calm air, it imposes an effective restraint against movements of the airplane relative to the air and causes violent movements of the airplane in rough air.

Law of Operating Control

The controlling moment in longitudinal control is exerted by means of the elevators, which are then to be moved according to some flight characteristic. Haus (reference 22) gives the following table of disturbance detectors and the quantities to which each is sensitive.

Instrument	Recorded quantity	Symbol
1. Air-speed indicator	Relative speed	U
2. Wind vane	Incidence	$\alpha = -w/u$
3. Free gyroscope, suspended at its c.g.	Absolute inclination	θ
4. Motor-driven gyroscope with precessional moment	Angular velocity	q
5. Pendulum or accelerometer along OX	Direction of apparent gravity	du/dt and $\sin\theta$
6. Accelerometer along OZ	Magnitude of apparent gravity	dw/dt and $\cos\theta$
7. Lift indicator	Magnitude of lift	iV^2 or uw
8. Rate-of-climb meter	Speed along vertical	w or $V \sin\theta$
9. Torsional accelerometer about OY	Angular acceleration	$\ddot{\theta}$

The elevators can be moved according to the indications of any of these instruments or combinations of them. The most successful controls, those of Sperry (references 23, 24, 25) and Smith (references 26, 27) are of type 3, operating according to the absolute inclination of the airplane in space. This type will therefore be considered in detail. Analysis of the other control types would be carried out in a manner similar to the one now presented.

Equations of Controlled Motion

In its broadest form, the type 3 control possesses velocity and acceleration components (types 4 and 9). A general solution including the displacement, the rate, and the acceleration components is no more difficult than that for the simple control, and the full form will therefore be considered.

The elevators are linked to the control through a servo mechanism, so that the pitching moment varies according to the displacement of the control. For small motions, a linear relation can be assumed and, for many controls, the assumption will also be valid for large displacements.

Since the control mechanism has inertia, an additional degree of freedom is introduced, and there are now four simultaneous equations of motion:

$$X = m(\dot{u} \quad Wq) \tag{27}$$

$$Z = m(\dot{w} - Uq) \tag{28}$$

$$M = B\dot{q} \tag{29}$$

$$F = m_c \ddot{\xi} \tag{30}$$

where F is force on the control.

m_c, effective inertia of the control referred to ξ.

ξ, displacement of the control.

The other symbols have their usual significance.

The full expressions for horizontal and vertical force are as usual, but equations (29) and (30) are now

$$B\dot{q} = u \frac{\partial M}{\partial u} + w \frac{\partial M}{\partial w} + q \frac{\partial M}{\partial q} - \xi \frac{\partial M}{\partial \xi} \tag{31}$$

$$m_c \ddot{\xi} = \theta \frac{\partial F}{\partial \theta} + q \frac{\partial F}{\partial q} + \ddot{\theta} \frac{\partial F}{\partial \ddot{\theta}} - \dot{\xi} \frac{\partial F}{\partial \dot{\xi}} - \xi \frac{\partial F}{\partial \xi} \qquad (32)$$

And, of course, $\xi \frac{\partial M}{\partial \xi}$ is the controlling moment corresponding to a control displacement ξ, and $\theta \frac{\partial F}{\partial \theta}$ is the force tending to operate the control for an angular displacement of the airplane θ. The remainder of the derivatives signify corresponding linkages.

Writing now the full determinant for the complementary solution where D is the operator d/dt,

$$\begin{vmatrix} D - X_u & -X_w & -g & 0 \\ -Z_u & D - Z_w & -g\gamma - DU_o & 0 \\ -M_u & -M_w & D^2 - DM_q & M_\xi \\ 0 & 0 & -D^2F_{\ddot{\theta}} - DF - F_\theta & D^2 + DF_{\dot{\xi}} + F_\xi \end{vmatrix} \qquad (33)$$

It has been assumed that most of the inertia of the control is effective after the three impulses have been combined to operate the elevators, which is equivalent to assuming that the error and the derivative controls have equal effective inertias and dampings. While experimental tests are required to determine the accuracy of this assumption, it is probably adequate for well-built controls, and controls could certainly be built for which it would hold exactly.

Note that in equation (33) the minor consisting of the first three rows and columns is the determinant for the uncontrolled airplane. Set the whole determinant equal to zero for the complementary solution and expand it in terms of this principal third order minor. This procedure gives

$$\Delta_o + \frac{M_\xi \left(D^2F_{\ddot{\theta}} + DF_q + F_\theta\right)}{D^2 + DF_{\dot{\xi}} + F_\xi} \begin{vmatrix} D - X_u & -X_w \\ -Z_u & D - Z_w \end{vmatrix} = 0 \qquad (34)$$

This form can be solved, but the final result will be more advantageous if it is put into nondimensional form. Going back to expression (33), write the dimensions of each term.

In terms of its dimensions, the determinant becomes

$$\begin{vmatrix} T^{-1} & T^{-1} & LT^{-2} & 0 \\ T^{-1} & T^{-1} & LT^{-2} & 0 \\ L^{-1}T^{-1} & L^{-1}T^{-1} & T^{-2} & L^{-1}T^{-2} \\ 0 & 0 & LT^{-2} & T^{-2} \end{vmatrix} \tag{35}$$

Following the same procedure as in nondimensionalizing the uncontrolled motion of the airplane, multiply the derivatives of

the first row by T

the second row by T

the third row by LT

the fourth row by T

the third column by $L^{-1}T$

the fourth column by T

The characteristic length of the dimensionless system is taken as L, the length of the tail moment arm. The characteristic time is defined by

$$T = m/(\rho/2\ SU) \tag{36}$$

On this basis, the unit of velocity is

$$L/T = U/\mu \tag{37}$$

where

$$\mu = m/(\rho/2\ SL) \tag{38}$$

and μ may be called the "relative density" of the airplane, being proportional to the mass and inversely proportional to the cube of the linear dimensions of the airplane. Glauert notes (reference 28) that apart from the derivative coefficients of the airplane, μ is the only parameter which affects the stability.

The dimensionless form of the derivatives is as dofined by Metcalf (reference 29) and other writers, but it is necessary to determine the form of the control derivatives.

In passing, it should be noted that another dimensionless system in use is based on the quantity ρ instead of $\rho/2$, as in the present paper. The mathematical expressions, however, are the same for both systems except for this one difference, and the numerical values of the derivatives differ only by a factor of 2.

Control Derivatives

It will be shown later that, for the purpose of this analysis, it is only necessary to determine the ideal control derivatives $m_{\theta, q, \ddot{\theta}}$.

The moment exerted by the tailplane is

$$M = C_{L_t} S_t \ \rho/2 \ U^2 L \tag{40}$$

where

C_{L_t} is tail lift coefficient.

S_t, tail area.

U, steady-flight airplane velocity.

L, tail moment arm.

Let δ, elevator angle.

Following the method of Koppen (reference 30) in nondimensionalizing the control derivatives, write

$$\frac{\partial M}{\partial \theta} = \left(\frac{\partial C_{L_t}}{\partial \delta}\right)\left(\frac{\partial \delta}{\partial \theta}\right)\frac{\rho}{2}\ S_t L U^2 \tag{41}$$

where $\dfrac{\partial C_{L_t}}{\partial \delta}$ is obtained from wind-tunnel data, and $\dfrac{\partial \delta}{\partial \theta}$ depends on the control-coupling ratios and sensitivity. Then

$$M_\theta T^2 = (\partial M/\partial \theta)\ (1/B) T^2 = \mu m_\theta \tag{42}$$

and $B = \eta m L^2$, where η is the "mass distribution factor" (reference 29) so that

$$m_\theta = \frac{1}{\eta}\left(\frac{\partial C_{Lt}}{\partial \delta}\right)\left(\frac{\partial \delta}{\partial \theta}\right)\frac{S_t}{S} \qquad (43)$$

Similarly

$$M_{qc}\,T = m_{qc} = \left(\frac{\partial C_{Lt}}{\partial \delta}\right)\left(\frac{\partial \delta}{\partial q}\right)\left(\frac{S_t}{S}\right)\frac{1}{\eta}\left(\frac{U}{L}\right) \qquad (44)$$

this factor is written m_{qc} to distinguish it from the natural m_q of the airplane. Finally

$$M_{\ddot{\theta}} = m_{\ddot{\theta}} = \left(\frac{\partial C_{Lt}}{\partial \delta}\right)\left(\frac{\partial \delta}{\partial \ddot{\theta}}\right)\frac{\left(\frac{\rho}{2}\,S_t\,U^2\,L\right)}{(\eta\,m\,L^2)} \qquad (45)$$

The f_θ, f_ξ, and m_ξ derivatives can be obtained in a similar manner, when it is desired to evaluate them in any particular problem.

The Dimensionless Determinant

The dimensionless form of the determinant becomes

$$\begin{vmatrix} D - x_u & -x_w & -\mu\,C_L & 0 \\ -z_u & D - z_w & -\mu(C_L\gamma + D) & 0 \\ -m_u & -m_w & D^2 - Dm_q & m_\xi \\ 0 & 0 & -D^2 f_{\ddot{\theta}} - Df_q - f_\theta & D^2 + Df_{\dot{\xi}} + f_\xi \end{vmatrix} \qquad (46)$$

and equation (34) becomes

$$\Delta_o + m_\xi\,\frac{D^2 f_{\ddot{\theta}} + Df_q + f_\theta}{D^2 + Df_{\dot{\xi}} + f_\xi}\begin{vmatrix} D - x_u & -x_w \\ -z_u & D - z_w \end{vmatrix} = 0 \qquad (47)$$

Call the second minor Δ_1. Divide the numerator and denominator of its multiplier by f_ξ. Then

$$\frac{\Delta_o}{\Delta_1} = \frac{\left(\frac{m_\xi}{f_\xi}\right)(D f_{\ddot{\theta}} + Df_q + f_\theta)}{D^2/f_\xi + Df_{\dot{\xi}}/f_\xi + 1} \tag{48}$$

Now

$$M_\theta = \frac{1}{B}\frac{\partial M}{\partial \theta}$$

$$M_\xi = \frac{1}{B}\frac{\partial M}{\partial \xi}$$

$$F_\theta = \frac{1}{m_c}\frac{\partial F}{\partial \theta}$$

$$F_\xi = \frac{1}{m_c}\frac{\partial F}{\partial \xi}$$

so that

$$M_\theta = M_\xi \frac{F_\theta}{F_\xi}$$

similarly

$$m_{\theta 1} = m_\xi \frac{f_\theta}{f_\xi} \tag{49}$$

The subscript $_1$ indicates that μ has not been extracted. This extraction could be very easily accomplished by proper evaluation of the other coefficients, such as making $\mu\ m_\xi$ take the place of m_ξ.

The numerator of equation (48) can now be written in terms of the ideal control derivatives, that is, the values which the control derivatives would have if there were no lag in the control. That is,

$$\left(\frac{m_\xi}{f_\xi}\right)(D^2 f_{\ddot{\theta}} + Df_q + \mu f_\theta) = D^2 m_{\ddot{\theta}} + Dm_{qc} + \mu\ m_\theta \tag{50}$$

Equation (48) can then be written as follows:

$$\Delta_o - \Delta_1(D^2 m_{\ddot{\theta}} + Dm_{qc} + \mu m_\theta) + \Delta_o\left(\frac{Df_{\dot{\xi}}}{f_\xi} + \frac{D^2}{f_\xi}\right) = 0 \tag{51}$$

Going back to equation (32), the control forces sum as

$$-m_c\ddot{\xi} + \theta\frac{\partial F}{\partial \theta} + q\frac{\partial F}{\partial q} + \ddot{\theta}\frac{\partial F}{\partial \ddot{\theta}} - \dot{\xi}\frac{\partial F}{\partial \dot{\xi}} - \xi\frac{\partial F}{\partial \xi} \tag{52}$$

and, when the error impulses are zero, the equation for the control alone is

$$m_c\ddot{\xi} + \dot{\xi}\frac{\partial F}{\partial \dot{\xi}} + \xi\frac{\partial F}{\partial \xi} = 0 \tag{53}$$

Compare this form with equation (5) for a generalized system of one degree of freedom.

$$\ddot{\xi} + 2\zeta\omega_n\dot{\xi} + \omega_n^2\xi = 0 \tag{54}$$

so that the F derivatives can be written in terms of the natural frequency and the damping of the isolated control. That is,

$$F_\xi = \omega_n^2 = (2\pi/T_n)^2 \tag{55}$$

$$F_{\dot{\xi}} = 2\zeta\omega_n = 2\zeta(2\pi/T_n) \tag{56}$$

Now $f_\xi = F_\xi T^2$, where T is the characteristic time of the airplane, and similarly

$$f_{\dot{\xi}} = F_{\dot{\xi}}T$$

so then

$$f_{\xi} = (2\pi\,T/T_n)^2 \tag{57}$$

$$f_{\dot{\xi}} = 2\zeta(2\pi\,T/T_n) \tag{58}$$

This form is convenient because T_n and ζ for the isolated control can be obtained by simple dynamic tests in the laboratory.

Substituting these values into equation (51) gives

$$\Delta_0 - \Delta_1(D^2m + Dm_{q_c} + \mu m_\theta) + \Delta_0\left[D^2(T_n/2\pi T)^2 + 2\zeta(T_n/2\pi T)D\right] = 0 \tag{59}$$

This division into three major terms is very convenient. If the control is nearly ideal, T_n is very small compared with T, and the quantity in the last bracket can be neglected, giving the equation for the ideally controlled airplane. If the airplane is uncontrolled, the second term is zero, leaving the original determinant for the uncontrolled motion.

Although it has not been so noted, during the nondimensionalizing, the operator D was replaced by the dimensionless operator DT, which was also written as D.

Effect of Ideal Control

Equation (59)is the complete form of the determinant for controlled longitudinal motion, an abridged form of which, neglecting the derivative components and control inertia, is given by Klemin (reference 3).

The effect of an ideal control on the period and the damping of the oscillation will first be determined. Assuming that T_n/T is sufficiently small so that the third term can be neglected, the first two terms of the equation can be expanded into the form,

$$aD^4 + bD^3 + cD^2 + dD + e = 0 \qquad (60)$$

Write $f = 1 - m_{\ddot{\theta}}$

$$m_{q1} = m_q + m_{qc}$$

where m_q is the natural damping in pitch of the airplane, and m_{qc} is the effective damping in pitch added by the first derivative component of the control.

It is necessary here to differentiate between the effect of m_q and of m_{qc} on the motion. Their effect in damping an oscillation once begun is equivalent. However, m_q is derived from the relative motion of the air and the tail surface primarily, whereas m_{qc} is derived by taking mechanically the first derivative of the angle of pitch snd using it to operate the elevators. Consequently, an airplane with large natural m_q may be expected to execute violent motion in rough air, in consequence of the restraint against motion relative to the gusts. On the other hand, an airplane heavily damped in pitch by means of the derivative control is restrained not relative to the air but to a set of fixed axes in space, determined by the gyros.

Protection from gusts, as far as the rotary damping is concerned, would then consist of replacement of the natural damping by artificial damping relative to space axes, as far as possible, with the limitation that the airplane must still be controllable manually in the event of failure of the automatic pilot.

Returning to equation (60), the coefficients are

$$
\begin{aligned}
a &= f \\
b &= -f(x_u+z_w) - m_{q1} \\
c &= f(x_u z_w - x_w z_u) + m_{q1}(x_u+z_w) - \mu m_w - \mu m_\theta \\
d &= -m_{q1}(x_u z_w - x_w z_u) + \mu m_w(x_u - C_L\theta) - \mu m_u(x_w + C_L) + \mu m_\theta(x_u+z_w) \\
e &= -C_L \mu m_w(z_u - x_u\theta) - C_L \mu m_u(x_w\theta - z_w) - \mu m_\theta(x_u z_w - x_w z_u)
\end{aligned}
\tag{61}
$$

The effect of m_{qc} upon the damping and the period is eqactly the same as the effect of m_q, and so it will be sufficient to note here that increasing the sensitivity of this component of the control will, over the normal-flight range, increase the damping and lengthen the period.

It is interesting to note that by making $m_{\ddot\theta} = 1.00$, f can be made zero. This method is equivalent to giving the airplane zero inertia in pitch. The quartic equation then reduces to a cubic.

As the m_θ control is the most widely used, it will be considered first.

Effect of Simple Displacement Control

Let $m_{\ddot\theta} = m_{qc} = 0$. Let the coefficients of the quartic represented by Δ_o be a_o, b_o, c_o, d_o, e_o. Then the effect of adding the m_θ derivative is to increase these coefficients so that

$$a = a_o$$

$$b = b_o$$

$$c = c_o - \mu m_\theta$$

$$d = d_o + \mu m_\theta (x_u + z_w)$$

$$e = e_o - \mu m_\theta (x_u z_w - x_w z_u) \qquad (62)$$

The effect of small values of m_θ on the long and the short oscillation has been calculated for two typical airplanes. Airplane 1 is the transport considered by Metcalf (reference 29) and Airplane 2 is a small 60-horsepower parasol monoplane treated by Soulé and Wheatley (reference 31).

Figure 3 gives the variation with m_θ of the period and the damping of the long oscillation. The effect of m_θ on the short oscillation in the range plotted is given by the following table for Airplane 2.

m_θ	Period (sec.)	Time to damp to ½ amplitude (sec.)
0	4.07	1.08
-.20	3.39	1.19
-.50	2.89	1.40

The effect of m_θ on the short oscillation is not very great. The period is shortened, as might be expected, since m_θ is a spring constant in pitch. The long oscillation shortens briefly and then, as its damping increases, lengthens its period, and finally becomes a pair of simple subsidences.

With complete restraint in pitch (m_θ infinite), the equation in D reduces to

$$D^2 - (x_u + z_w)\, D + \begin{vmatrix} x_u & z_u \\ x_w & z_w \end{vmatrix} = 0 \qquad (63)$$

The success of the simple displacement control in practice is explained by the tremendous increase in the damping of the long oscillation. The time to damp to $\frac{1}{2}$ amplitude can, in the case considered, be reduced from 20 to 2 seconds. For an ideal control, then, both the long and the short oscillation would disappear almost at once after a disturbance.

Moreover, the values of m_θ used in the foregoing example have been extremely conservative. Klemin (reference 3), investigating this type of control, presents $M_\theta = -2,160$ as a practical value for the airplane he considers and, in dimensionless form, this value is equivalent to $m_\theta = -900$, approximately. Klemin finds, in consequence, that the long oscillation disappears completely and is replaced by a pair of subsidences.

It should be understood that the control does not create any new damping. It simply makes a more economical use of the damping that is already available in the system. The poor distribution of damping between the two oscillations is well known and the m_θ control may be considered as a sort of equalizing valve, which allows some of the damping of the short oscillation to flow into the long oscillation. The simple displacement control is allowed, therefore, by the peculiar original condition of the system, to produce an effect comparable with that of an error and derivative control. Needless to say, this fact is highly advantageous from considerations of mechanical simplicity of the control.

Use of the Acceleration Component

The controlling moment can also be made to depend on the second derivative of the angular displacement, which is equivalent to increasing or decreasing the effective inertia about the lateral axis of the airplane.

The inertia might be increased ($m_{\ddot{\theta}}$ negative) to reduce the initial pitching acceleration under the influence of gusts, or it might be decreased to permit a more rapid damping of the subsequent motion. Simple increase of the effective inertia will not be ordinarily tolerated, however, because of its unfavorable effect on the stability of the motion.

Now suppose a positive value of $m_{\ddot{\theta}}$ were chosen. A simple form results if $m_{\ddot{\theta}}$ is taken as 1.00, as this choice provides that the airplane have zero effective inertia in pitch. The coefficients of the reduced equation then bear the relation to the original equation that

$$a = 0$$

$$b = m_{q1}$$

$$c = c_o - (x_u z_w - x_w z_u) = \text{very nearly } c_o$$

$$d = d_o$$

$$e = e_o$$

A few trial calculations show that $m_{\ddot{\theta}}$ affects the short oscillation almost exclusively, which might be expected on noting that only the first two coefficients are greatly changed by $m_{\ddot{\theta}}$. A very good approximate factorization, since b is large, is

$$(D + b)\ (D^2 + (bc - d)/b^2\ D + d/b) = 0 \qquad (64)$$

and the short oscillation becomes a heavily damped subsidence, with the long oscillation practically unchanged. But the short oscillation was already satisfactorily well damped. Therefore, the use of a positive acceleration component of the control does not seem to be justified, and the introduction of a negative $m_{\ddot{\theta}}$ component would be satisfactory only if it were selective, operating only to oppose movement away from equilibrium.

Introduction of Control Lag

Actual controls frequently exhibit a fast residual oscillation, which one writer (reference 27) describes as the effect of the control trying to act upon the short oscillation. More exactly, this new oscillation is probably caused by the additional degree of freedom supplied by the control, and certainly depends on control inertia.

In order to investigate the nature of this residual oscillation, the full form of equation (59) will now be used, except that the two derivative drives will be neglected, because they are not at present incorporated in the Sperry or the Smith controls. The equation is then

$$\Delta_o - \Delta_1 \mu m_\theta + \Delta_o \left[D^2 (T_n/2\pi T)^2 + 2\zeta (T_n/2\pi T) D \right] = 0 \qquad (65)$$

Let

$$r = T / T_n 2\pi \qquad (66)$$

Let the coefficients of Δ_o be a_o, b_o, c_o, d_o, e_o and the coefficients of Δ_{o1} (the ideally controlled determinant) be a_1, b_1, c_1, d_1, e_1.

Then, writing equation (65) as the sextic

$$aD^6 + bD^5 + cD^4 + dD^3 + eD^2 + fD + g = 0 \qquad (67)$$

the coefficients of the sextic are, in terms of the coefficients of the ideally controlled and the uncontrolled motion,

$$a = r^2$$

$$b = 2\zeta r + r^2 b_o$$

$$c = 1 + 2\zeta r b_o + r^2 c_o$$

$$d = b_1 + 2\zeta r c_o + r^2 d_o$$

$$e = c_1 + 2\zeta r d_o + r^2 e_o$$

$$f = d_1 + 2\zeta r e_o$$

$$g = e_1 \qquad (68)$$

The effect of increasing the natural period and the damping of a control for a given airplane and a given static linkage m_θ can therefore be carried through in an orderly manner by the use of the relations given in equations (68). This procedure has been carried out for a typical case.

One of the ideal cases treated was for $m_\theta = -0.50$. This case has been extended to include control inertia and damping. It is very easy to write down the sextic, but its solution is not an enjoyable task, although there are methods of accomplishing the solution. It is much easier simply to determine the effect on the stability of the motion of allowing the control inertia to become finite.

Effect of Inertia Lag on Stability

The simple quartic for the uncontrolled motion was

$$D^4 + 4.20\ D^3 + 11.96\ D^2 + 1.94\ D + 1.30 = 0$$

and when $m_\delta = -0.50$, this expression becomes

$$D^4 + 4.20\ D^3 + 20.96\ D^2 + 19.40\ D + 7.70 = 0$$

This equation is for the airplane of reference 31, and the derivatives were based on ρ instead of $\rho/2$. The damping ratio of the control was held constant at $\zeta = 0.20$ while the natural period T_n was increased from zero. The stability changes were followed by both Routh's discriminant for the sextic, and the use of the simultanaous equations in natural frequency. The results have already been given in figure 2. The airplane becomes unstable when the natural period of the control exceeds 1.94 seconds.

Effect of Damping in the Control

In order to determine the effect of control damping on stability, the definitely unstable case of $T_n = 2.40$ seconds was reconsidered with the damping ratio of the control increased to 1.00, that is, critical damping. Routh's discriminant then became positive and equal to 1,275. Increasing the damping then restores stability or, in other words, postpones the critical point of neutral stability.

Although it may seem contrary to common sense to improve the stability by making the control act less rapidly, it will be remembered that the uncontrolled airplane was stable, and adding an infinite amount of damping to the control can do no worse than restore it to this state. If the airplane were originally unstable, damping might eventually have an adverse effect.

As in the constant-speed control, where the effect has been discussed in detail (reference 32), damping in the control allows the initial surge error to be somewhat larger on account of the sluggishness of the control, but successive error surges are reduced because the tendency of the control to overshoot has been curbed.

Damping is not completely beneficial, but it can compensate for the worst property of controls, which is inertia lag. Stability of the motion is attained by damping; good performance is attained by reducing control inertia.

Solution of the Critical Case

When the natural period of the control is 1.94 seconds, the motion becomes critically stable, and an unending oscillation is present. From the intersection of the two curves for the frequency, the frequency of this oscillation is $\omega_n^2 = 19$. Two of the roots of the sextic are then known, and the sextic can be factored into

$$(D^2 + 19)\ (D^4 + 6.45D^3 + 35D^2 + 37D + 15) = 0$$

The quartic can be solved by Zimmerman's method

$$(D^2 + 19)\ (D^2 + 1.20D + 0.52)\ (D^2 + 4.25D + 28.8) = 0$$

Compare this with the factorization for the ideal controlled airplane with no control inertia

$$(D^2 + 1.15D + 0.455)\ (D^2 + 3.05D + 16.9) = 0$$

The long oscillation has hardly been affected. It seems justifiable, then, to obtain a rough factorization of the sextic in any case below the critical by dividing through by the quadratic factor of the ideal quartic, which corresponds to the long oscillation.

The short oscillation is also not greatly modified. It is the new, or "residual" oscillation introduced by the control that becomes unstable. Because of the characteristics of the linkages, the residual oscillation apparently becomes unstable when its period approaches that of the short oscillation.

The period of the residual oscillation shortens and its damping improves as the inertia of the control is reduced; the oscillation vanishes completely for an ideal control.

Determination of Control Frequency and Damping

The convenience of the representation of the control as a single degree of freedom system is seen when the laboratory procedure necessary to obtain T_n, ζ, and m_θ is outlined. The complete control, with dummy control surfaces suitably weighted to represent their equivalent inertia in flight, is mounted on a test platform so that it can be easily rotated (in pitch for a longitudinal control). The platform is given a very sudden change in inclination, and the response of the control is recorded by means of a pencil or equivalent recording means, attached to an output member, such as a control push rod of the mechanism. From the record, which may show a damped oscillation or an aperiodic approach to the new position, the damping ratio and the natural frequency of the control can be determined by elementary vibration formulas.

The ideal control derivative m_θ is determined by noting the steady-state control angle for a given angular displacement of the disturbance indicator of the control. Given this ratio, m_θ is determined from equation (43).

Only three quantities are required, and they are the three that express the effect of the control on the motion of the airplane. Therefore, they also serve as convenient means of comparing one control against another of the same type.

Methods of obtaining advantageous values of the three parameters remain in the province of detailed control design, but the parameters offer a means of determining the suitability of an existing control for a particular airplane.

Suppression of Disturbances in Gusts

Insuring a rapid decadence of motions once begun is only half the job of a successful automatic control. The other half consists of the reduction of initial error surges from the desired course, as the airplane encounters an external disturbance.

The magnitude of the surge error can be investigated with the aid of operational calculus or by means of mechanical methods of solution, such as the differential analyzer.

The initial acceleration of the airplane under the influence of various kinds of gust can be written for each of the three degrees of freedom as follows,

Gust	Initial acceleration	Derivative depends on
Vertical - w_o	$x_w w_o$	Induced drag
	$z_w w_o$	Aspect ratio
	$m_w w_o$	Static stability
Head or Tail - u_o	$x_u u_o$	Total drag
	$z_u u_o$	Lift
	$m_u u_o$	Power application
Rotary gust - q_o	$x_q q_o$	
	$z_q q_o$	
	$m_q q_o$	Tail size and efficiency

*These two are negligible.

In order to reduce the effect of the gust in any case, the corresponding derivative should be made small. If the derivative is zero, the airplane will not be affected by a gust in that sense. Because the motions in each degree of freedom are related, however, an airplane with, say, m_u = zero would develop a pitching motion in response to a head gust, but the original forcing function would be applied only as a vertical and a horizontal force.

The only derivatives that can be modified appreciably by the designer are the static stability m_w and the damping m_q. Experimental data (reference 33) indicate that, in general, airplanes with short period and heavy damping do the most pitching in rough air, corresponding roughly to large m_w and large m_q.

The initial accelerations of the airplane represented by $m_w w_o$ and $m_q q_o$ can then be reduced, and the stability of the airplane will not be impaired if the reduced derivatives are augmented by m_θ and m_{qc} supplied by an automatic pilot. The value of m_θ is not numerically equivalent to m_w but has a similar effect, being, in fact, a much more desirable derivative as it allows a more satisfactory distribution of damping, both m_q and m_{qc}, between the two oscillations.

Systematic computations by Haus (reference 34) shows that, even in the absence of the automatic pilot, reduction of m_w reduces the violence of the motion.

Unfortunately, little can be done to reduce the initial response of the airplane to a vertical gust. For the first second or so, the motion is given very closely by

$$w/w_o = 1 - e^{z_w t/T} \tag{69}$$

and z_w, depending on aspect ratio, cannot be reduced appreciably. This situation might have been foreseen for, if the air supporting the airplane rises, the airplane itself must modify its course in space.

Once started, however, the disturbed motion can be made to disappear much more rapidly with the aid of an automatic pilot, as Klemin has shown (reference 3) by a number of calculations. Klemin also shows that the vertical motion in response to a head gust is very much eased by an m_θ control. Wilson, in an early paper on the effect of gusts (reference 35), also indicated the beneficial effects of a complete constraint in pitch.

Conclusions

On the basis of the material presented, some general conclusions can be drawn concerning the operation of a control sensitive to the angle of pitch and its derivatives. In the absence of complete experimental data, the conclusions must be presented with the reservation that, while they satisfy available data, very little data are available.

1. The control with gyroscopic references gives the airplane sensitivity with respect to axes fixed in space.

2. The simple pitch control readjusts the proportions of the system, so that the available damping is more equally distributed between the long and the short oscillation, with the result that the long oscillation can be made to disappear as rapidly as the short oscillation and to take on the form of two simple subsidences.

3. The benefit of the simple partial restraint in pitch, therefore, is derived from the ability of the displacement control to act as an equivalent rate control.

4. The addition of an actual rate control increases the damping in pitch (m_q) of the airplane, without increasing the sensitivity $(q\ m_q)$ to a rotary gust.

5. There is no advantage in using an unselective second derivative control and not much advantage in using a selective control.

6. Inertia in the control introduces a third oscillation which can be mistaken for the short oscillation when it becomes troublesome.

7. The third oscillation becomes unstable when the inertia of the control is increased beyond a critical value determined by the airplane characteristics and the control damping ratio.

8. Damping in the control reduces the effectiveness of the control, but stabilizes the residual oscillation.

9. Flight in rough air will be improved (greater course stability) by reducing the static stability (m_w) and the natural damping in pitch (m_q), and by adding an automatic control to supply sensitivity to angle of pitch (m_θ) and absolute damping in pitch (m_{qc}).

CONTROLLED LATERAL MOTION

The lateral motion of the airplane for small disturbances consists of translation along the Y axis (sideslip) and rotations about the X and Z axes (rolling and yawing). The uncontrolled motion is represented by an equation of the fifth degree, of which one of the roots is zero, signifying that the airplane is insensitive to direction in azimuth. In the remaining quartic, one of the roots is very large compared with the remaining three, and is very nearly equal to the damping in roll. That is to say, the resistance of the airplane to rate of roll is so large that the rolling subsidence preserves its character in spite of the other lateral motions of the airplane. The rolling motion disappears almost at once in normal flight and may give trouble only at the stall.

Of the remaining three roots, two are conjugate complex and one is real, defining an oscillation and a subsidence (or divergence). The only airplane characteristics influencing the roots at the disposal of the designer are the amount of dihedral and the amount of fin and rudder area. Generally speaking, a large amount of static stability causes the real root to become negative (spiral instability) while an excess of negative static stability beyond a small minimum value will cause the oscillation to become first of increasing magnitude and then to separate into a rapidly increasing exponential mode (reference 36).

Controlled Lateral Motion

The lateral motion is controlled by the rudder about the yawing axis and by the ailerons about the rolling axis. The ailerons, as a rule, in addition to exerting a rolling moment, will also apply a yawing moment, usually of opposite sign to the direction of the desired turn and of the order of a tenth the rolling moment.

The rudder control, if actuated by azimuth indications, makes up for the natural deficiency of the airplane in azimuth. But the provision of a sense of direction does not guarantee adequate damping of the motion. The use of an angle of yaw control is in this respect not so fortunate as the addition of an angle of pitch control for the longitudinal motion. The rolling subsidence remains very rapid, the original short-period oscillation is sensibly unchanged (reference 6), but the slow spiral divergence or subsidence formerly present has now become a long-period banking and yawing oscillation which may be poorly damped and which depends on the dihedral of the airplane for the regulation of its period, of the order of 15 to 20 seconds (reference 27).

The motion can be modified by operating the rudder or the ailerons according to other characteristics of the motion.

Law of Operating Control

Haus (reference 37) has given the following table of disturbance detectors, according to the indications of which the ailerons or rudder can be operated.

	Instrument	Recorded quantity	Symbol
1.	Vane with vertical axis	Angle of sideslip	v/u
2.	Free gyroscope	Yaw with respect to axes fixed in space	ψ
3.	Free gyroscope	Roll with respect to axes fixed in space	φ
4.	Gyroscope producing precessional couple	Angular velocity of rolling	p
5.	Gyroscope, or difference in linear speed of wing tips	Angular velocity of yawing	r
6.	Pendulum in ZOY plane, or accelerometer along OY	Direction of apparent gravity	$g \sin \varphi + dv/dt + Vr$
7,8	Torsional accelerometers about X and Z axes	Angular acceleration about OX and OZ	$\ddot{\varphi}, \ddot{\psi}$
9.	Compass	Yaw with respect to earth's magnetic field	ψ

It may be stated as fundamental that the primary purpose of the lateral control is to give the airplane sensitivity in azimuth. In this respect, it differs from the longitudinal control that operated to improve course stability already present. Secondary control components are then used to improve the resulting motion.

The Sperry control operates the ailerons according to angle of bank to improve the motion, and the Smith control employs both aileron and a component of rudder motion proportional to angle of bank. The Askania control moves the rudder according to both angle and rate of yaw. Garner has shown (reference 6) that all these secondary controls affect the damping of the oscillations. The rate of yaw control, however, affects the short oscillation almost exclusively instead of the long-period "course" oscillation which it was intended to improve. The aileron controls improve the motion by modifying the distribution of damping although they are simple displacement controls.

Lag in the Control

There are two separate controls, as a rule, for the rudder and the ailerons. Consequently, two additional equations of motion are introduced, one for each new degree of freedom, and the determination of the complete motion will require the solution of a ninth-order equation.

It is possible that the highest order terms may be neglected, or that lag in the aileron control may be neglected compared with lag in the rudder control, because of the much greater damping in roll than in yaw. Sufficient calculations are not yet available, however, to endorse such simplifications.

Following the method of introducing control inertia and damping that was treated in detail under longitudinal control, the complete equations of controlled lateral motion will now be presented, for an airplane with rudder and ailerons moved according to angles and first and second derivatives of yaw and roll. This arrangement is analogous to the control provided by the Sperry Gyropilot, except for the addition of the derivative components. Controls using other laws of operation can be similarly analyzed.

Equations of Motion

For simplicity, the airplane will be considered in level flight so that $\theta_o = 0$, and the product of inertia E will be considered small enough to be neglected. The equations of lateral motion are then

$$Y = m(dv/dt+Ur) \tag{70}$$

$$L = A\ dp/dt \tag{71}$$

$$N = C\ dr/dt \tag{72}$$

Where Y is lateral force, L is rolling moment, and N is yawing moment.

There are two separate controls, one for the ailerons, and one for the rudder. Let

F, force on aileron control.

T, force on rudder control.

ξ, displacement of aileron control.

η, displacement of rudder control.

Force derivatives for the controls are obtained in the same manner as force and moment derivatives for the airplane. Thus,

$$F_\xi = (1/m_{control})\ (\partial F/\partial \xi) \quad \text{etc.} \tag{73}$$

Then the free motion of each control, with no forcing function or external force acting can be written as

$$\text{Aileron control} \quad (D^2 + DF_{\dot\xi} + F_\xi)\ \xi = 0 \tag{74}$$

$$\text{Rudder control} \quad (D^2 + DT_{\dot\eta} + T_\eta)\ \eta = 0 \tag{75}$$

These two equations define the natural period and the damping of each control.

The full set of simultaneous equations for the complementary solution of the motion of the controlled airplane is given by

Side force

$$(D-Y_v)\ v + (-Y_\varphi)\ \varphi + (DU)\ \psi + 0 + 0 = 0 \tag{76}$$

Rolling moment

$$(-L_v\ v + (D^2-DL_p)\varphi + (-DL_r)\psi + (-L_\xi)\ \xi\ 0 \tag{77}$$

Yawing moment

$$(-N_v)v + (-DN_p)\varphi + (D^2-DN_r)\psi + (-N_\xi)\xi + (-N_\eta)\eta = 0 \tag{78}$$

Aileron control force

$$0 + (-D^2F_{\ddot\varphi}-DF_p-F_\varphi)\varphi + 0 + (D^2+DF_\xi+F_\xi)\xi + 0 = 0 \tag{79}$$

Rudder control force

$$0 + 0 + (-D^2T_{\ddot\psi}-DT_r-T_\psi)\psi + 0 + (D^2+DT_\eta+T_{\dot\eta})\eta = 0 \tag{80}$$

Dimensions of Determinant

It is again desired to have the derivatives dimensionless. The dimensions of the determinant resulting from the foregoing equations are

$$\begin{vmatrix} T^{-1} & LT^{-2} & LT^{-2} & 0 & 0 \\ L^{-1}T^{-1} & T^{-2} & T^{-2} & L^{-1}T^{-2} & 0 \\ L^{-1}T^{-2} & T^{-2} & T^{-2} & L^{-1}T^{-2} & L^{-1}T^{-2} \\ 0 & LT^{-2} & 0 & T^{-2} & 0 \\ 0 & 0 & LT^{-2} & 0 & T^{-2} \end{vmatrix} \tag{81}$$

In order to make this expression dimensionless, multiply

the first line by T

the second and third lines by TL

the second and third columns by T/L

the fourth row and column by T

the fifth row and column by T

The unit of time is again $T = m/(\rho/2\ SU)$, but the unit of length according to Koppen's notation is the wing semispan $(b/2)$.

Expansion of the Determinant

As in the longitudinal case, the determinant can be expanded in terms of ideal control derivatives and the control frequencies and the damping ratios, so that it will not be necessary here to evaluate the coupling terms L_ξ, N_ξ, N_η, etc.

Certain minors of the full determinant will be used sufficiently often to justify a general symbol. Let the dimensionless determinant for the uncontrolled airplane be Δ_o. Then

$$\Delta_o = \begin{vmatrix} d - y_v & \mu\, C_L & d\mu \\ -l_v & d^2 - dl_p & -dl_r \\ -n_v & -dn_p & d^2 - dn_r \end{vmatrix} \tag{82}$$

Also let

$$\Delta_1 = \begin{vmatrix} d - y_v & d\mu \\ -l_v & -dl_r \end{vmatrix} \tag{83}$$

$$\Delta_2 = \begin{vmatrix} d - y_v & d\mu \\ -n_v & d^2 - dn_r \end{vmatrix} \tag{84}$$

$$\Delta_3 = \begin{vmatrix} d - y_v & \mu\, C_L \\ -l_v & d^2 - dl_p \end{vmatrix} \tag{85}$$

Now also assume the shorthand notation that

$$l_{\xi F} = \frac{D^2\, l_{\ddot{\varphi}} + Dl_{pc} + l_\varphi}{D^2 (T_{n2}/2\pi T)^2 + D2\zeta_2 (T_{n2}/2\pi T) + 1} \tag{86}$$

$$n_{\eta F} = \frac{D^2\, n_{\ddot{\varphi}} + Dn_{pc} + n_\varphi}{D^2 (T_{n2}/2\pi T)^2 + D2\zeta_2 (T_{n2}/2\pi T) + 1} \tag{87}$$

$$n_{\eta T} = \frac{D^2\, n_{\ddot{\varphi}} + Dn_{rc} + n_{\psi}}{D^2(T_{n_1}/2\pi T)^2 + D2\zeta_1(T_{n_1}/2\pi T) + 1} \tag{88}$$

Note that, if the yawing moment of the ailerons is assumed to be proportional to the rolling moment over the normal range of aileron angles,

$$n_{\xi F} = k\ l_{\xi F} \tag{89}$$

where k is a constant of proportionality.

In terms of these minors and fractions, the fifth-order determinant can now be expanded into a form more convenient for treatment.

Expanded Form of the Determinant

The expansion is of the form

uncontrolled airplane component + aileron yaw component + aileron roll component + rudder yaw component + rudder yaw aileron roll side force component = 0 (90)

Written symbolically, this equation becomes

$$\Delta_0 + n_{\xi F}\,\Delta_1 - l_{\xi F}\,\Delta_2 - n_{\eta T}\,\Delta_3 + n_{\eta T}\,l_{\xi F}(D-y_v) = 0 \tag{91}$$

It is interesting to note that, when the rudder is moved according to the angle of roll, the effect corresponds to the yawing action of the ailerons and is therefore included in the second term of equation (91). Thus, rudder movement according to this relation can be made to balance exactly the adverse yawing moment of the ailerons. The operation is one that human pilots perform instinctively.

Effect of Ideal Controls

It is interesting to compare the effects of the various components of control outlined in the foregoing on the motion of the airplane. If the simple displacement controls of the rudder and ailerons are perfectly quick and powerful, the airplane is completely restrained in roll and yaw, and equation (91) reduces to simply

$$D - y_v = 0 \qquad (92)$$

which denotes a subsidence in sideslip, determined by the side force produced by sideslipping.

The uncontrolled motion is represented by the determinant Δ_o which, when expanded, gives an equation of the form

$$D^5 + bD^4 + cD^3 + dD^2 + eD = 0 \qquad (93)$$

The effect of the control derivatives is to increase or decrease these coefficients or to raise them to higher powers of D. Suppose, first, that the controls are ideal, and that they are simple displacement controls with no derivative components. Then the additions to the coefficients can be tabulated.

Add to coefficient	Rudder moment proportional to yaw	Rudder moment proportional to angle of bank	Aileron yawing moment proportional to angle of bank	Aileron rolling moment proportional to angle of bank
b	0	0	0	0
c	$-\mu n_\psi$	0	0	$-\mu l_\varphi$
d	$\mu n_\psi(l_p + y_v)$	$-\mu n_\varphi(l_r)$	$-\mu n_\varphi(l_r)$	$\mu l_\varphi(n_r + y_v)$
e	$-\mu n_\psi(l_p y_v)$	$\mu n_\varphi(l_r y_v + \mu l_v)$	$\mu n_\varphi(l_r y_v + \mu l_v)$	$-\mu l_\varphi(n_r y_v + \mu n_v)$
f	$-\mu n_\psi(\mu l_v C_L)$	0	0	0

(94)

When both rudder displacement proportional to yaw and aileron displacement proportional to angle of bank are present, there is an additional cross-product term

$$\mu n_\psi \ \mu l_\varphi(D - y_v) \qquad (95)$$

Discussion

In order that there be course stability, f must be greater than zero. If a rudder control is used alone, then for f to be positive l_v must be greater than zero numerically (negative in this system of units; l_v is the derivative depending on dihedral and with l_v zero (slight negative dihedral), the pilot cannot maintain course stability without the aid of the ailerons. Addition of the simple aileron control adds a term from (95) proportional to y_v, the side force in sideslip, and this term increases the course static stability.

If f were zero, e would determine the spiral stability of the airplane. For most airplanes, e is negative, indicating a divergence. If a simple aileron control is added alone, the positiveness of e can be increased and the airplane made spirally stable regardless of a small deficiency of dihedral.

When the rudder control is added, the airplane cannot be spirally unstable, and the term e becomes indicative of, but not the criterion of, the damping of the course oscillation. The damping is evidently improved by large dihedral, side area, and control derivative l_φ.

None of these simple controls affects b, which very closely represents the rolling subsidence. Therefore, this component of the motion will be substantially unchanged by the controls in normal flight.

A satisfactory approximate factorization has not yet been developed for the short oscillation, but its damping is probably improved by a large value of coefficient c.

The effect of the first and the second derivative components in the controls is to raise the individual terms of (94) by one and two rows, respectively, and to add them to the original simple displacement components. That is, in the first column, the n_ψ control contributes $-\mu n_\psi$ to the c coefficient. An n_{rc} component will contribute $-\mu n_{rc}$ to the b term. And so on with the remaining terms.

More detailed conclusions will require extensive calculations.

Nonideal Controls

There is no difficulty in expanding equation (91), when the controls do not have negligible inertia. Neglecting the higher derivative components of the control, which are treated in a similar manner, and letting

$$D^2 (T_{n_1}/2\pi T)^2 + D2\zeta_1 (T_{n_1}/2\pi T) = C_1, \text{ etc.} \tag{96}$$

Equation (91) expands to

$$\begin{aligned}(C_1+1)(C_2+1)\,\Delta_o + (C_1+1)\,n_\varphi\Delta_1 - (C_1+1)\,l_\varphi\Delta_2 - \\ - (C_2+1)\,n_\psi\Delta_3 + n_\psi l_\varphi (D-y_v) = 0\end{aligned} \tag{97}$$

or, when separated into ideal and nonideal components:

Ideal $\quad \Delta_o + n_\varphi\Delta_1 - l_\varphi\Delta_2 - n_\psi\Delta_3 + n_\psi l_\varphi (D-y_v)$

First order contribution $\quad +(C_1+C_2)\Delta_o + C_1(n_\varphi\Delta_1 - l_\varphi\Delta_2) - C_2 n_\psi\Delta_3$

Second order contribution $\quad + C_1\, C_2\, \Delta_o = 0 \qquad (98)$

When either the rudder or the aileron control is being considered alone, the frequency expression simplifies considerably. In its full form as given, it involves a ninth-order equation, but $C_1 C_2$ is probably very small compared with the first-order terms, and possibly may be neglected.

The stability of the motion can then be investigated by means of the natural-frequency expressions for the septic.

January 18, 1939.

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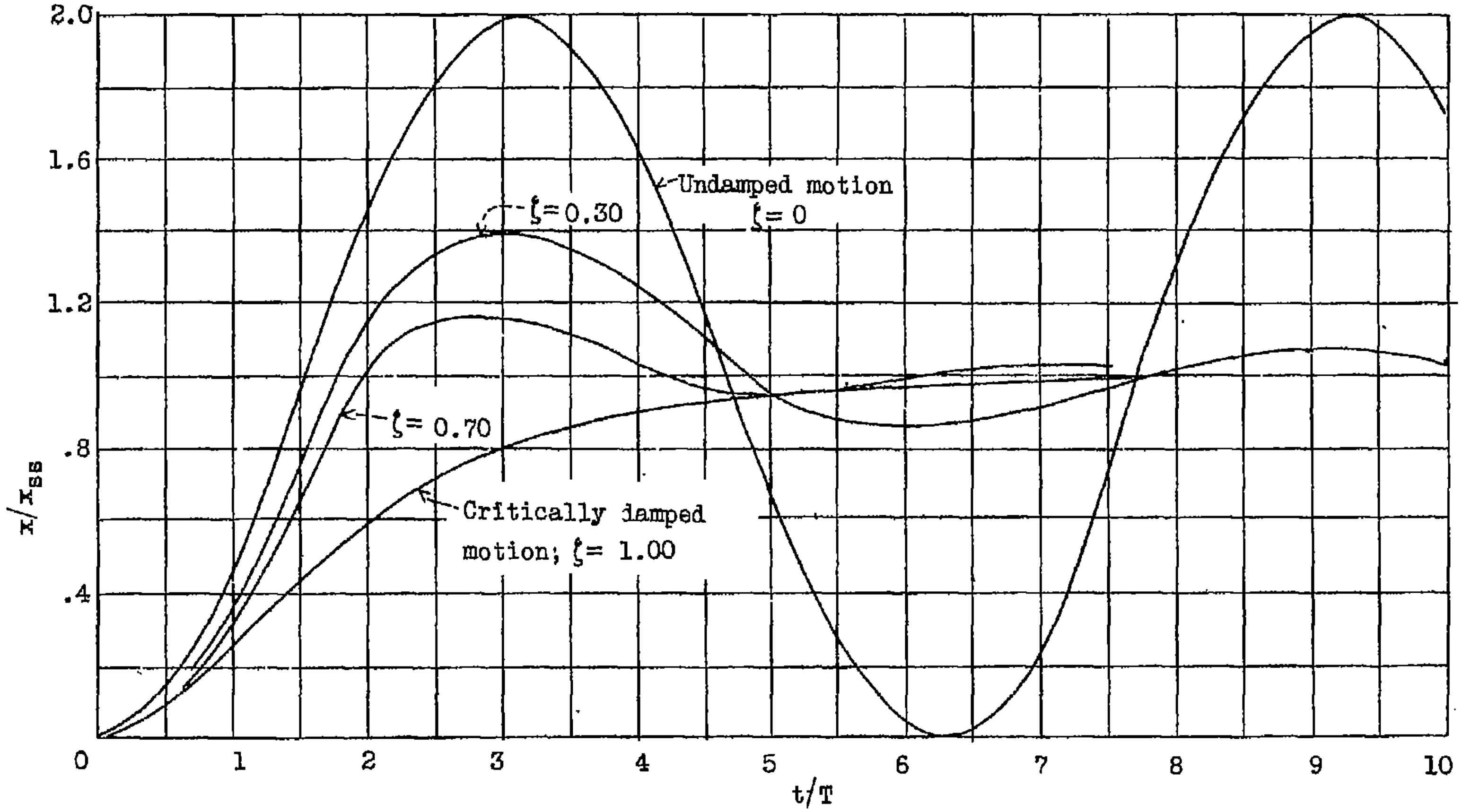

Figure 1.- Effect of damping on response of system of one degree of freedom.

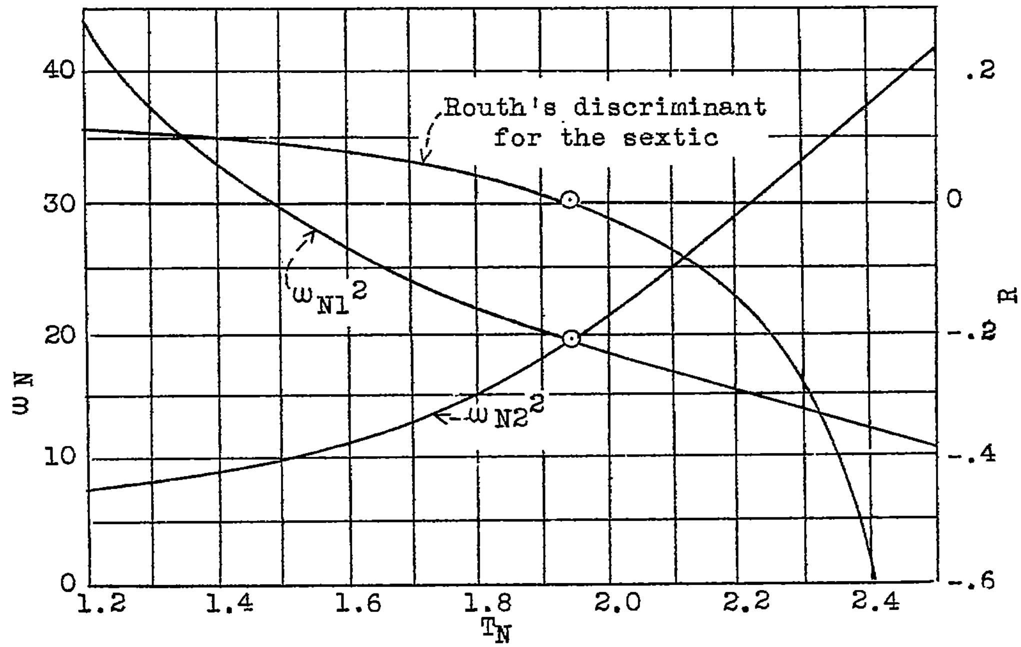

Figure 2.- Comparison of methods for determining stability of sextic equation.

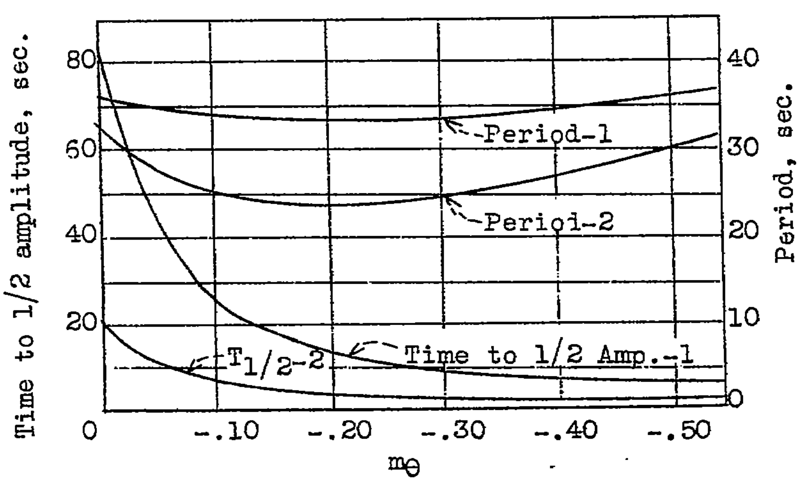

Figure 3.- Effect of displacement control on period and damping of phugoid.